

On New Moment Estimation of Parameters of the Generalized Gamma Distribution Using It's Characterization

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Abstract

In this paper, the three estimators for three parameters of the generalized gamma distribution are proposed by using its characterization, and shown to be more convenient and more efficient than the maximum likelihood estimator and the moment estimator for small samples. Furthermore, the estimators for the parameter of gamma and Weibull distributions are obtained. Finally, the previous results of an example in the reliability is compared with the estimators mentioned above.

1. Introduction

In early work with the generalized gamma distribution there was difficult in developing inference procedures (e.g. Stacy and Mihram, 1965., Parr and Webster, 1965; Harter 1967; Hager and Bain, 1970), but work by Prentice (1974) clarified matters a great deal and its inference can now be fairly easily handled (e.g. Farewell and Prentice, 1977; Lawless, 1980). Much of the difficulties with the model arise because the generalized gamma distribution with very different sets of parameter values look alike. Therefore Prentice (1974) considered the distribution in a reparameterized form that reduces this effect and makes properties of the distribution much more transparent. In this paper, we use its characterization to derive new estimator of parameter of the generalized gamma distribution. Furthermore, the comparison with previous results is discussed.

2. New moment estimator for parameters of the generalized gamma distribution

For deriving new moment estimators for three parameters of the generalized gamma distribution, we need the following theorem obtained by using the similar approach of Hwang and Hu (1999).

Theorem 2.1: Let $n \geq 3$ and let X_1, X_2, \dots, X_n be n positive i.i.d. random variables having a probability density function $f(x)$. Then the independence of the sample mean \overline{X}_n and the sample coefficient of variation $V_n = S_n / \overline{X}_n$ is equivalent to that $f(x)$ is a generalized gamma density where S_n is

the sample standard deviation.

This Theorem can be used to derive the expectation and the variance of $V_n^2 = (S_n / \overline{X_n})^2$, where $\overline{X_n}$ and S_n denote respectively the sample mean and the sample standard deviation through this paper.

Theorem 2.2: Let $n \geq 3$ and let X_1, X_2, \dots, X_n be n i.i.d. random samples drawn from a population having a generalized gamma density

$$g(x; \lambda, \beta, k) = \frac{\lambda \beta}{\Gamma(k)} (\lambda x)^{k\beta-1} \exp[-(\lambda x)^\beta] \quad x > 0, \lambda > 0, \beta > 0, k > 0.$$

Then

$$E\left(\frac{S_n^2}{\overline{X_n}^2}\right) = \frac{n \cdot \left[\Gamma(k) \cdot \Gamma(k + \frac{2}{\beta}) - \Gamma^2(k + \frac{1}{\beta}) \right]}{\Gamma(k) \cdot \Gamma(k + \frac{2}{\beta}) + (n-1) \cdot \Gamma^2(k + \frac{1}{\beta})}$$

For a set of n i.i.d. random samples X_1, X_2, \dots, X_n , the following equalities can be easily to prove

$$E(\overline{X_n}) = \frac{\Gamma(k + \frac{1}{\beta})}{\lambda \Gamma(k)}, E\left(\frac{X_1^\beta + X_2^\beta + \dots + X_n^\beta}{n}\right) = \frac{k}{\lambda^\beta}$$

Based on Theorems 2.2 and two equalities above, we set, by using moment estimation approach, three equations for finding three estimators of parameters β, λ, k respectively as follows:

$$\frac{S_n^2}{n \overline{X_n}^2} = \frac{\Gamma(k) \Gamma(k + \frac{2}{\beta}) - \Gamma^2(k + \frac{1}{\beta})}{\Gamma(k) \Gamma(k + \frac{2}{\beta}) + (n-1) \Gamma^2(k + \frac{1}{\beta})},$$

$$\overline{X_n} = \frac{\Gamma(k + \frac{1}{\beta})}{\lambda \Gamma(k)}, \quad \sum_{i=1}^n X_i^\beta = \frac{nk}{\lambda^\beta}$$

Simplifying first equality further, we have

$$c \Gamma^2(k + \frac{1}{\beta}) = \Gamma(k + \frac{2}{\beta}) \Gamma(k)$$

where $c = \frac{n \overline{X_n}^2 + (n-1) S_n^2}{n \overline{X_n}^2 - S_n^2}$. Thus the solutions for β, λ, k obtained by solving the three equations

simultaneously are proposed for their estimators presented respectively by $\hat{\beta}, \hat{\lambda}, \hat{k}$.

When the generalized gamma distribution reduces to the gamma distribution ($\beta = 1$), we have

$$\hat{\lambda} = \frac{k}{x_n}, \quad \hat{k} = \frac{\overline{x_n^2}}{S_n^2} - \frac{1}{n}$$

which are same as those proposed by Hwang and Huang (2002), while to the Weibull distribution ($k=1$) the proposed two estimators can be obtained by solving the following two equations

$$\lambda \beta \overline{X_n} = \Gamma\left(\frac{1}{\beta}\right), \quad c\Gamma^2\left(\frac{1}{\beta}\right) = 2\beta \Gamma\left(\frac{2}{\beta}\right)$$

3. The comparison with previous results

The comparison of our estimators with maximum likelihood estimators has been done in terms of mean square error by using the simulation for gamma and Weibull distribution; our estimators are more reliable than MLE's for $n \geq 25$. Note that the result for gamma distribution has been presented by Hwang and Huang (2002).

More statistical investigations using these new three estimators have been compared with the previous results which had been done by Lieblein and Zelen (1956), and Chang (2000), and the significant results are found.

References

- Stacy, E.W. (1962). A generalization of the gamma distribution. *Ann. Math. Stat.*, 33 1187-1192.
- Parr, V.B., and J.T. Webster (1965). A method for discriminating between failure density functions used in reliability predictions. *Technometrics*, 7, 1-10.
- Harter, H.L. (1964). Maximum-likelihood estimation of the parameters of a four-parameter generalized gamma population for complete and censored samples. *Technometrics*, 9, 159-165.
- Harger, H.W., and L.J. Bain (1970). Inferential procedures for the generalized gamma distribution. *J. Am. Stat. Assoc.*, 65, 1601-1609.
- Prentice, R.L. (1974). A log gamma model and its maximum likelihood estimation. *Biometrika*, 61, 539-544.

- Farewell, V.T., and R.L. Prentice (1977). A study of distribution shape in life testing. *Technometrics*, 19, 69-75.
- Lawless, J.F. (1980). Inference in the generalized gamma and log gamma distributions. *Technometrics*, 22, 409-419.
- Hwang, T.Y., and C.Y. Hu (1999). On a characterization of the gamma distribution: the independence of the sample mean and the sample coefficient of variation. *Annals of the Institute of Statistics Mathematics*, 51, 749-753, Japan.
- Hwang, T.Y., and P.H. Huang (2002). On new moment estimation of parameters of the gamma distribution using it's characterization. *Annals of the Institute of Statistics Mathematics*, 54, 840-847, Japan.
- Lieblein, J. and M. Zelen (1956). Statistical investigation of the fatigue life of deep groove ball bearings. *J. Res. Not. Bur. Stand.*, 57, 273-316.
- Chang, K.I.(2002). On the model of the life of deep groove ball Bearing. *Master thesis, Institute of Statistics, National Tsing Hua University, Taiwan.*